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LETTER TO THE EDITOR

Counterintuitive behaviour in games based on spin models

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Abstract. A large class of games based on spin models is analysed. For these games, the long-time gain can be calculated exactly. It is shown that the mixing of two losing strategies may lead to a winning one, but also that the mixing of two winning ones may lead to a loss. Also, the mixing of a losing and a winning strategy may give unexpected results. This behaviour is due to two general features: (i) the class of games is such that mixing the playing of two games is equivalent to playing a third one, and (ii) the break-even boundaries for these games are curved.

Recently [1, 2], it has been pointed that there are games such that if each is played singly it is certain that there is a loss in the long run, whereas playing the games randomly a net gain results (the 'Parrondo paradox'). In this letter, a class of games (including those considered in [1]) is introduced for which several kinds of counterintuitive behaviour can be proved to occur. For these games, the probabilities of winning or losing one unit depend on the accumulated total gain G (which can be a loss for $G < 0$). The precise definition is as follows: if the accumulated gain G is congruent to l mod M, then there is the probability a_l of winning ($G \to G + 1$) and $1 - a_l$ of losing ($G \to G - 1$). This is a generalization of the games considered previously [1] to full cyclic symmetry (with period M). These games are patterned after the spin models known as M-state clock models [3]. For such a spin model, the interaction energy between two spins in states i and j is a function of $i - j$ mod M only. In the special case $d = a_0$, $a = a_1 = a_2 = \cdots = a_{M-1}$, these are related to the Potts [4] (for $M \geqslant 3$) or Ising [5] (for $M = 2$) models. These spin models have only two different interaction energies, the 'diagonal' one (corresponding to d) being realized if both spins are in the same state and the 'nondiagonal' one (corresponding to a) if they are in different states. One could similarly introduce games with the symmetry of the Zamolodchikov–Monastyrskii [6] or any other model with a symmetry group containing a cyclic subgroup [3]. Since the probabilities only depend on the value of G achieved in the previous round, the playing of such a game consecutively is a Markov process, which can be described by a discrete master equation for $P_l(n, k)$, the probability that G has the value $nM + l$ after k rounds:

$$\begin{aligned} P_0(n, k+1) &= a_{M-1}P_{M-1}(n-1, k) + (1-a_1)P_1(n, k) \\ P_l(n, k+1) &= a_{l-1}P_{l-1}(n, k) + (1-a_{l+1})P_{l+1}(n, k) \qquad 1 \leqslant l \leqslant M-2 \\ P_{M-1}(n, k+1) &= a_{M-2}P_{M-2}(n, k) + (1-a_0)P_0(n+1, k). \end{aligned} \tag{1}$$

This master equation is linear in the probabilities a_l, so that mixing the play of a game A with probability p with the play of a game B with parameters b_l and probability $1 - p$ amounts to the consecutive playing of a game C with parameters $c_l = pa_l + (1 - p)b_l$. This property will be called *closure with respect to convex mixing*. The special cases treated in [1] correspond

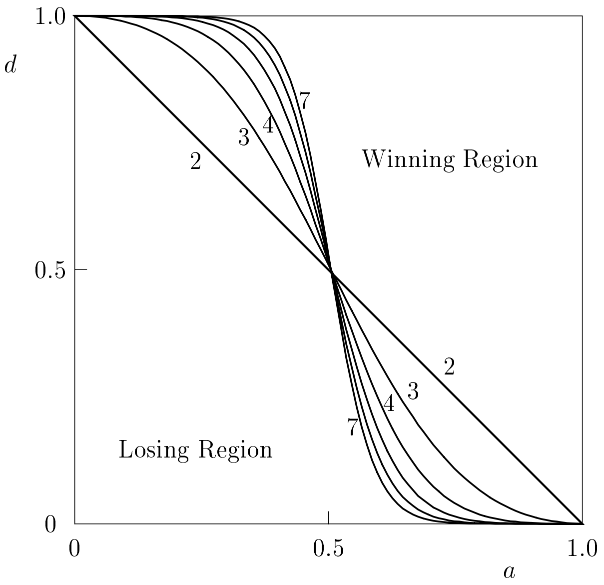

Figure 1. The break-even curves for the Ising ($M = 2$) and Potts games (for $3 \leqslant M \leqslant 7$). Here d is the 'diagonal' probability of winning for the case $G = 0 \bmod M$, $a = a_1 = \cdots = a_{M-1}$ the 'off-diagonal' one for $G \neq 0 \bmod M$.

to the Potts model for $M = 3$: one of the games (A) is chosen so that $a = d$ holds; the other one (B) is general.

The master equation can be solved exactly to give an expression for the average value of G, the accumulated gain:

$$\langle G\rangle(k+1) = \langle G\rangle(k) + \sum_{l=0}^{M-1}(2a_l - 1)q_l(k) \tag{2}$$

where $q_l(k) = \sum_{n=-\infty}^{\infty} P_l(n,k)$ and the initial conditions are $\langle G\rangle(0) = 0$ and $q_l(0) = \delta_{l,0}$. In the long run, ergodicity implies that the additive term in equation (2) can be evaluated with q_l such that these form the eigenvector with eigenvalue 1 of the (sparse) matrix of equation (1). The break-even boundary can then easily be found to be given by

$$a_0 a_1 \ldots a_{M-1} = (1 - a_0)(1 - a_1)\cdots(1 - a_{M-1}). \tag{3}$$

These break-even boundaries are shown in figure 1 for the Ising and Potts cases for $M \leqslant 7$. Since here one has $d = a_0$ and $a = a_1 = \cdots = a_{M-1}$, these curves are given by

$$d = (1-a)^{M-1}/[a^{M-1} + (1-a)^{M-1}]. \tag{4}$$

Except for the Ising case, these are all curves with convex and concave parts. They interpolate between the Ising case and the limit $M \to \infty$, in which case again a straight line ($a = \frac{1}{2}$) results. This fact, together with the closure of these games with respect to convex mixing, now accounts for the following counterintuitive situations:

(i) Mixing two winning games results in a straight line, which may intersect the losing region. This is shown in figure 2 for the Potts $M = 3$ case as the line W_1–W_2.
(ii) Mixing two losing games may result in a line intersecting the winning region. In figure 2 this is, for example, the line L_1–L_2.
(iii) Mixing a losing game (L_2) with a winning one (W_2) may result in a loss–gain–loss–gain sequence; this is the dashed line in figure 2.

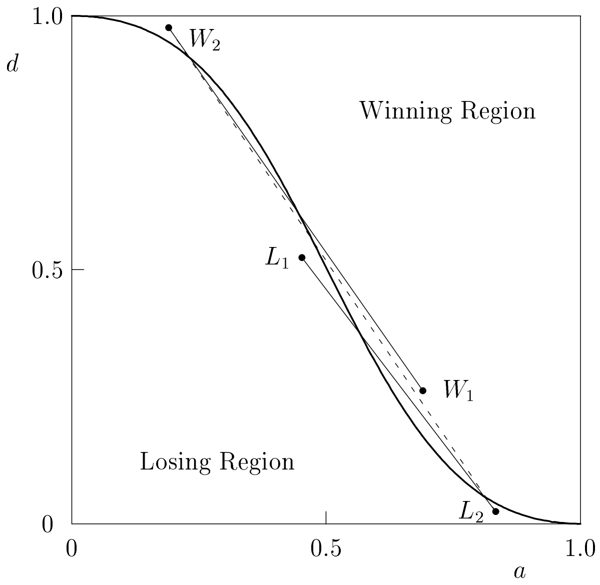

Figure 2. An illustration of the three counterintuitive possibilities for the Potts case with $M = 3$.

In all cases, the straight lines obtained by mixing the games are given by varying the probability p as described above; they are given explicitly by the d-coordinate $c_0(p) = pa_0 + (1 - p)b_0$ as a function of $c(p) = pa + (1 - p)b$ as a-coordinate, where $c(p) = c_1(p) = \cdots = c_{M-1}(p)$, $a = a_1 = \cdots = a_{M-1}$, $b = b_1 = \cdots = b_{M-1}$, for fixed (a_0, a) (the game of the first end point) and (b_0, b) (the game of the second end point).

It is actually possible to calculate the average gain g per game exactly for this type of model (in the limit of many games played). For the Potts model cases with $M = 3$ or 4, this is, for example, given by

$$
\begin{aligned}
g &= \frac{3[a^2 d - (1-a)^2(1-d)]}{2 + a^2 d + (1-a)^2(1-d)} \qquad M = 3 \\
g &= \frac{2[a^3 d - (1-a)^3(1-d)]}{1 + (1-a-d)(1-2a)} \qquad M = 4.
\end{aligned}
\tag{5}
$$

To give an idea of the amounts involved, this has been plotted in figure 3 for the three possibilities listed above for $M = 3$. Especially in the case of figure 3(c), the intermediate absolute gains are much larger than those of the original games.

The authors of [1] tried to explain the ‘Parrondo paradox’ by means of an analogy to a flashing Brownian ratchet [7]. In view of the above exact results, such an analogy does not seem to be necessary. In fact, as soon as one goes away from the situation in [1], where one of the games lies on the line $a = d$, this analogy would imply the interplay of two such ratchets, which does not provide a simple picture.

Although in this letter a particular, exactly solvable type of game has been considered, it should be clear that there may be many situations for which results similar to cases (i)–(iii) above may occur. It will certainly always be the case if the mixed playing of two games may be interpreted as leading to a game which is a convex linear combination of the original ones *and* if the break-even boundary for this type of game is curved. Especially, case (iii) may, in a symmetric situation (player 1 has a strategy given by parameters a_l, player 2 with parameters $1 - a_l$, see equation (3)) be interpreted as a zero-sum two-person game [8]: each player would like his strategy to be adopted, since it is a winning game for him (losing for the other player).

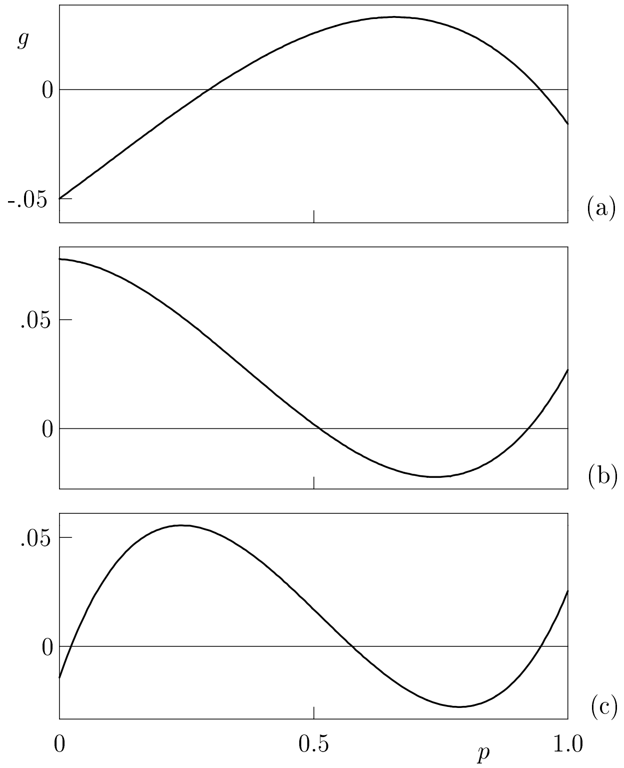

Figure 3. The total gain per game for the three cases: (*a*) W_1–W_2, (*b*) L_1–L_2 and (*c*) L_2–W_2 for the Potts case with $M = 3$.

If the players adopt an exact compromise, both break even, but a small deviation from this point (in the direction of player 1's winning strategy, say) may result in a huge gain for player 2 (and a corresponding loss for player 1). These counterintuitive results could have far-reaching consequences for games of economic [8], evolutionary [9] or even political importance, since they may be seen as dangers resulting from a compromise. If this compromise is not exact, it may seem satisfactory to both parties, but is extremely unjust for one of them in practice.

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